Monads and Quantitative Equational Theories for Nondeterminism and Probability

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Abstract

The monad of convex sets of probability distributions is a well-known tool for modelling the combination of nondeterministic and probabilistic computational effects. In this work we lift this monad from the category of sets to the category of extended metric spaces, by means of the Hausdorff and Kantorovich metric liftings. Our main result is the presentation of this lifted monad in terms of the quantitative equational theory of convex semilattices, using the framework of quantitative algebras recently introduced by Mardare, Panangaden and Plotkin.

2012 ACM Subject Classification Theory of computation → Logic and verification; Theory of computation → Axiomatic semantics; Theory of computation → Categorical semantics

Keywords and phrases Computational Effects, Monads, Metric Spaces, Quantitative Algebras

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2020.28

Funding Both authors have been partially supported by the French project ANR-16-CE25-0011 REPAS. Vignudelli has been partially supported by the European Research Council (ERC) under the European Union's Horizon 2020 programme (CoVeCe, grant agreement No 678157) and by the LABEX MILYON (ANR-10-LABX-0070) of Université de Lyon, within the program "Investissements d'Avenir" (ANR-11-IDEX-0007) operated by the French National Research Agency (ANR).

Acknowledgements The authors are grateful to the anonymous reviewers and to the authors of [7] for their useful comments and suggestions.

1 Introduction

In the theory of programming languages the categorical concept of *monad* is used to handle computational effects [43, 44]. As main examples, the *powerset monad* ($\mathcal{P}$) and the *probability distribution monad* ($\mathcal{D}$) are used to handle nondeterministic and probabilistic behaviours, respectively. It is of course desirable to handle the combination of these two effects to model, for instance, concurrent randomised protocols where nondeterminism arises from the action of an unpredictable scheduler and probability from the use of randomised procedures such as coin tosses. However, the composite functor $\mathcal{P} \circ \mathcal{D}$ is not a monad (see, e.g., [52]).

A well–known way to handle this technical issue is to use instead the *convex powerset of distributions monad* ($\mathcal{C}$) which restricts $\mathcal{P} \circ \mathcal{D}$ by only admitting sets of probability distributions that are closed under the formation of *convex combinations* (see [50, 29, 28, 42, 41, 33, 39] and Section 2). Restricting $\mathcal{P} \circ \mathcal{D}$ to $\mathcal{C}$ is not only mathematically convenient, because it leads to a monad, but also natural as convexity captures the possibility of a scheduler to make probabilistic choices, as originally observed by Segala [46]. Suppose indeed that a scheduler can select between two probabilistic behaviours $\{d_1, d_2\}$ for execution. It is reasonable to assume that said scheduler can also, with the aid of a (biased) coin, choose d_1 with probability p and d_2 with probability $1 - p$. Hence, effectively, the scheduler can choose any behaviour in $\{p \cdot d_1 + (1 - p) \cdot d_2 \mid p \in [0, 1]\}$, which is indeed a convex set of distributions.

31st International Conference on Concurrency Theory (CONCUR 2020).
Editors: Igor Konnov and Laura Kovács; Article No. 28; pp. 28:1–28:18
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

In a recent work [13] the authors provide a proof for the following result: the equational theory $\mathtt{Th}_{CS}$ of convex semilattices is a *presentation* of the **Set** monad $\mathcal{C}$. This means (see Section 2 for details) that the category $\mathbf{A}(\mathtt{Th}_{CS})$ of convex semilattices and their homomorphisms is isomorphic to the category $\mathbf{EM}(\mathcal{C})$ of Eilenberg-Moore algebras for $\mathcal{C}$.

Presentation results of this kind have a number of applications in computer science due to the interplay between the structure (syntax) and the dynamics (behaviour) of systems. For example, it follows from the presentation result of [13] that the free convex semilattice with set of generators X is isomorphic to $\mathcal{C}(X)$. This allows us to manipulate elements of $\mathcal{C}(X)$ as convex semilattice terms modulo the equations of $\mathtt{Th}_{CS}$ and, similarly, to perform equational reasoning steps using facts (e.g., from geometry) related to the mathematical structure of $\mathcal{C}(X)$. Applications in the field of program semantics and concurrency theory arise by combining coalgebraic reasoning methods, associated with the use of monads as behaviour functors, and algebraic methods, which are made available by presentation theorems. Well known examples include *bisimulation up–to techniques* (e.g., up–to congruence [11]) and the categorical approach to structural operational semantics, introduced by Turi and Plotkin in [51] (see also [35]) and based on the notion of *bialgebras*.

The category **EMet**, having extended metric spaces as objects and non–expansive maps as morphisms, is a natural mathematical setting[1] which can replace the category **Set** when it is desirable to switch from the concept of *program equivalence* to that of *program distance*. This has been a very active topic of research in the last two decades (see, e.g, [45, 27, 15, 23, 16]). In this context, it is necessary to deal with monads on **EMet**. Variants of the **Set** monads $\mathcal{P}$ and $\mathcal{D}$ have been proposed on **EMet** (see, e.g., [15, 8] and Section 3), and are technically based on different types of *metric liftings*, due to Hausdorff and Kantorovich.

Contributions of this work. In this work we investigate a **EMet** variant of the **Set** monad $\mathcal{C}$, which we denote by $\hat{\mathcal{C}}$. As a functor, $\hat{\mathcal{C}} : \mathbf{EMet} \to \mathbf{EMet}$ maps a metric space (X, d) to the metric space $(\mathcal{C}(X), HK(d))$, the collection of non–empty, finitely generated convex sets of finitely supported probability distributions on X endowed with the metric $H(K(d))$, the Hausdorff lifting of the Kantorovich lifting of the metric d.

$$\hat{\mathcal{C}} : \mathbf{EMet} \to \mathbf{EMet} \qquad (X, d) \mapsto \Big(\mathcal{C}(X), H(K(d))\Big).$$

As a first contribution, in Section 4 we give a direct proof of the fact that $\hat{\mathcal{C}}$ is indeed a monad on **EMet**. This result does not seem straightforward to prove. Most notably, establishing the non–expansiveness of the monad multiplication $\mu^{\hat{\mathcal{C}}}$ requires some detailed calculations.

Our second and main result concerns the presentation of the **EMet** monad $\hat{\mathcal{C}}$. Presentations of monads in **Set** are given in terms of categories of algebras (in the sense of universal algebra) and their homomorphisms, but these are not adequate in the metric setting. For this reason we use, instead, the recently introduced apparatus of *quantitative algebras* and *quantitative equational theories* of [36] (see also [37, 7, 5, 4]). This framework generalises that of universal algebra and equational reasoning by dealing with quantitative algebras, which are metric spaces equipped with non–expansive operations over a signature, and quantitative equations of the form $s =_\epsilon t$, intuitively expressing that the distance between terms s and t is less than or equal to ϵ. In Section 4 we define the quantitative equational theory $\mathtt{QTh}_{CS}$ of

[1] The category **EMet** of extended metric spaces carries additional categorical structure compared to the category **Met** of ordinary metric spaces such as, e.g., the existence of coproducts. This structure is often useful in the field of program semantics. All the results obtained in this paper can be easily adapted to hold in the category **Met**.

quantitative convex semilattices, and in Section 5 we prove the presentation result (Theorem 36): the category $\mathbf{EM}(\hat{\mathcal{C}})$ of Eilenberg-Moore algebras for $\hat{\mathcal{C}}$ is isomorphic to the category $\mathbf{QA}(\mathtt{QTh}_{CS})$ of quantitative convex semilattices and their non–expansive homomorphims.

Relation with other works. This work continues the research path opened in the seminal [36] (see also subsequent works [37, 7, 5, 4]) where the authors investigated the connection between the quantitative theories of semilattices ($\mathtt{QTh}_{SL}$) and convex algebras ($\mathtt{QTh}_{CA}$) and the monads $\hat{\mathcal{P}}$ and $\hat{\mathcal{D}}$, which are **EMet** variants of $\mathcal{P}$ and $\mathcal{D}$, respectively. Hence, our work constitutes a natural step forward. From a technical standpoint, there is a difference between our main presentation result and those of [36] regarding $\mathtt{QTh}_{SL}$ and $\mathtt{QTh}_{CA}$ (corollaries 9.4 and 10.6 respectively in [36]). Indeed, in [36] the authors only provide representations of the free objects in the categories $\mathbf{QA}(\mathtt{QTh}_{SL})$ and $\mathbf{QA}(\mathtt{QTh}_{CA})$. While this suffices in many applications, we believe that proving a full presentation, in the sense introduced and investigated in this work, provides a more general and useful result, giving a representation for the whole categorical structure and not just for free objects. This said, the technical machinery developed in [36] suffices, with minor additional work[2], to establish the following presentation results in our sense: $\mathbf{QA}(\mathtt{QTh}_{SL}) \cong \mathbf{EM}(\hat{\mathcal{P}})$ and $\mathbf{QA}(\mathtt{QTh}_{CA}) \cong \mathbf{EM}(\hat{\mathcal{D}})$.

2 Monads on Sets and Equational Theories

In this section we present basic definitions and results regarding monads. We assume the reader is familiar with the basic concepts of category theory (see [3] as a reference).

▶ **Definition 1.** *Given a category* $\mathbf{C}$*, a monad on* $\mathbf{C}$ *is a triple* $(\mathcal{M}, \eta, \mu)$ *composed of a functor* $\mathcal{M}\colon \mathbf{C} \to \mathbf{C}$ *together with two natural transformations: a unit* $\eta\colon id \Rightarrow \mathcal{M}$*, where* id *is the identity functor on* $\mathbf{C}$*, and a multiplication* $\mu\colon \mathcal{M}^2 \Rightarrow \mathcal{M}$*, satisfying the two laws* $\mu \circ \eta\mathcal{M} = \mu \circ \mathcal{M}\eta = id$ *and* $\mu \circ \mathcal{M}\mu = \mu \circ \mu\mathcal{M}$*.*

We now introduce three relevant monads on the category **Set** of sets and functions.

▶ **Definition 2.** *The* non–empty finite powerset *monad* $(\mathcal{P}, \eta^{\mathcal{P}}, \mu^{\mathcal{P}})$ *on* **Set** *is defined as follows. Given an object* X *in* **Set***,* $\mathcal{P}(X) = \{X' \subseteq X \mid X' \neq \emptyset \text{ and } X' \text{ is finite}\}$*. Given an arrow* $f : X \to Y$*,* $\mathcal{P}(f) : \mathcal{P}(X) \to \mathcal{P}(Y)$ *is defined as* $\mathcal{P}(f)(X') = \bigcup_{x \in X'} f(x)$ *for any* $X' \in \mathcal{P}(X)$*. The unit* $\eta^{\mathcal{P}}_X : X \to \mathcal{P}(X)$ *is defined as* $\eta^{\mathcal{P}}_X(x) = \{x\}$*, and the multiplication* $\mu^{\mathcal{P}}_X : \mathcal{P}\mathcal{P}(X) \to \mathcal{P}(X)$ *is defined as* $\mu^{\mathcal{P}}_X(\{X_1, \dots, X_n\}) = \bigcup_{i=1}^n X_i$*.*

A probability distribution on a set X is a function $\Delta : X \to [0, 1]$ such that $\sum_{x\in X} \Delta(x) = 1$. The *support* of Δ is defined as the set $supp(\Delta) = \{x \in X \mid \Delta(x) \neq 0\}$. In this paper we only consider probability distributions with finite support which we often just refer to as distributions. The Dirac distribution $\delta(x)$ is defined as $\delta(x)(x') = 1$ if $x' = x$ and $\delta(x)(x') = 0$ otherwise. We often denote a distribution having $supp(\Delta) = \{x_1, x_2\}$ using the expression $p_1x_1 + p_2x_2$, with $p_i = \Delta(x_i)$. Analogously, we let $\sum_{i=1}^n p_ix_i$ denote a distribution Δ with support $\{x_1, \dots, x_n\}$ and with $p_i = \Delta(x_i)$.

▶ **Definition 3.** *The* finitely supported probability distribution *monad* $(\mathcal{D}, \eta^{\mathcal{D}}, \mu^{\mathcal{D}})$ *on* **Set** *is defined as follows. For objects* X *in* **Set***,* $\mathcal{D}(X) = \{\Delta \mid \Delta \text{ is a finitely supported probability distribution on } X\}$*. For arrows* $f : X \to Y$ *in* **Set***,* $\mathcal{D}(f) : \mathcal{D}(X) \to \mathcal{D}(Y)$ *is defined as*

[2] The proof structure of our Theorem 36 can be adapted (and in fact much simplified due to the simpler nature of $\mathtt{QTh}_{SL}$ and $\mathtt{QTh}_{CA}$ compared to $\mathtt{QTh}_{CS}$) to obtain these isomorphisms of categories. The recent result [7, Thm 4.2] might also provide an alternative proof method.

$\mathcal{D}(f)(\Delta) = \big(y \mapsto \sum_{x\in f^{-1}(y)} \Delta(x)\big)$. The unit $\eta^{\mathcal{D}}_X : X \to \mathcal{D}(X)$ is defined as $\eta_X(x) = \delta(x)$. The multiplication $\mu^{\mathcal{D}}_X : \mathcal{D}\mathcal{D}(X) \to \mathcal{D}(X)$ is defined, for $\sum_{i=1}^n p_i\Delta_i \in \mathcal{D}\mathcal{D}(X)$, as $\mu^{\mathcal{D}}_X(\sum_{i=1}^n p_i\Delta_i) = \big(x \mapsto \sum_{i=1}^n p_i \cdot \Delta_i(x)\big)$.

▶ Remark 4. Given elements $\Delta_1, \dots, \Delta_n \in \mathcal{D}(X)$, the expression $\sum_{i=1}^n p_i\Delta_i$ denotes an element in $\mathcal{D}\mathcal{D}(X)$. The set $\mathcal{D}(X)$ can be seen as a convex subset of the real vector space $\mathbb{R}^X$, so in order to avoid confusion with the notation $\sum_{i=1}^n p_i\Delta_i$ we will use the following dot–notation $\sum_{i=1}^n p_i \cdot \Delta_i$ to denote convex combinations of distributions: $\sum_{i=1}^n p_i \cdot \Delta_i = \mu^{\mathcal{D}}_X(\sum_{i=1}^n p_i\Delta_i) = \big(x \mapsto \sum_{i=1}^n p_i \cdot \Delta_i(x)\big)$. Hence, $\sum_{i=1}^n p_i\Delta_i$ denotes an element of $\mathcal{D}\mathcal{D}(X)$ (a distribution of distributions), while $\sum_{i=1}^n p_i \cdot \Delta_i$ denotes an element of $\mathcal{D}(X)$.

Given a collection $S \subseteq \mathcal{D}(X)$ of distributions, we can construct its *convex closure* $cc(S) = \{\sum_{i=1}^n p_i \cdot \Delta_i \mid n \geq 1, \Delta_i \in S \text{ for all } i, \text{ and } \sum_{i=1}^n p_i = 1\}$. Note that $cc(cc(S)) = cc(S)$. A subset $S \subseteq \mathcal{D}(X)$ is *convex* if $S = cc(S)$. We say that a convex set $S \subseteq \mathcal{D}(X)$ is *finitely generated* if there exists a finite set $S' \subseteq \mathcal{D}(X)$ (i.e., $S' \in \mathcal{P}\mathcal{D}(X)$) such that $S = cc(S')$. Given a finitely generated convex set $S \subseteq \mathcal{D}(X)$, there exists one minimal (with respect to the inclusion order) finite set $\mathtt{UB}(S) \in \mathcal{P}\mathcal{D}(X)$ such that $S = cc(\mathtt{UB}(S))$. The finite set $\mathtt{UB}(S)$ is referred to as the *unique base* of S (see, e.g., [14]). The distributions in $\mathtt{UB}(S)$ are convex–linear independent, i.e., if $\mathtt{UB}(S) = \{\Delta_1, \dots, \Delta_n\}$, then for all i, $\Delta_i \notin cc(\{\Delta_j \mid j \neq i\})$.

▶ **Definition 5.** *The* finitely generated non-empty convex powerset of distributions *monad* $(\mathcal{C}, \eta^{\mathcal{C}}, \mu^{\mathcal{C}})$ *on* **Set** *is defined as follows. Given an object X in* **Set***, $\mathcal{C}(X)$ is the collection of non-empty finitely generated convex sets of finitely supported probability distributions on X, i.e., $\mathcal{C}(X) = \{cc(S) \mid S \in \mathcal{P}\mathcal{D}(X)\}$. Given an arrow $f : X \to Y$ in* **Set***, the arrow $\mathcal{C}(f) : \mathcal{C}(X) \to \mathcal{C}(Y)$ is defined as $\mathcal{C}(f)(S) = \{\mathcal{D}(f)(\Delta) \mid \Delta \in S\}$. The unit $\eta^{\mathcal{C}}_X : X \to \mathcal{C}(X)$ is defined as $\eta^{\mathcal{C}}_X(x) = \{\delta(x)\}$, the singleton (convex) set consisting of the Dirac distribution. The mutiplication $\mu^{\mathcal{C}}_X : \mathcal{C}\mathcal{C}(X) \to \mathcal{C}(X)$ is defined, for any $S \in \mathcal{C}\mathcal{C}(X)$, as $\mu^{\mathcal{C}}_X(S) = \bigcup_{\Delta\in S} \mathtt{WMS}(\Delta)$, where, for any $\Delta \in \mathcal{D}\mathcal{C}(X)$ of the form $\sum_{i=1}^n p_iS_i$, with $S_i \in \mathcal{C}(X)$, the* weighted Minkowski sum *operation $\mathtt{WMS} : \mathcal{D}\mathcal{C}(X) \to \mathcal{C}(X)$ is defined as $\mathtt{WMS}(\Delta) = \{\sum_{i=1}^n p_i \cdot \Delta_i \mid \text{for each } 1 \leq i \leq n,\ \Delta_i \in S_i\}$.*

2.1 Equational Theories and Monad Presentations

An important concept regarding monads is that of algebras for a monad.

▶ **Definition 6.** *Let $(\mathcal{M} : \mathbf{C} \to \mathbf{C}, \eta, \mu)$ be a monad. An algebra for $\mathcal{M}$ is a pair (A, h) where $A \in \mathbf{C}$ is an object and $h : \mathcal{M}(A) \to A$ is a morphism such that: $h \circ \eta_A = id_A$ and $h \circ \mathcal{M}h = h \circ \mu_A$. Given two $\mathcal{M}$–algebras (A, h) and (A', h'), a $\mathcal{M}$–*algebra morphism *is an arrow $f : A \to A'$ in $\mathbf{C}$ such that $f \circ h = h' \circ \mathcal{M}(f)$. The category of Eilenberg-Moore algebras for $\mathcal{M}$, denoted by $\mathbf{EM}(\mathcal{M})$, has $\mathcal{M}$–algebras as objects and $\mathcal{M}$–morphisms as arrows.*

The definitions above are purely categorical and, as a consequence, the category $\mathbf{EM}(\mathcal{M})$ is sometimes hard to work with as an abstract entity. It is therefore very useful when $\mathbf{EM}(\mathcal{M})$ can be proven isomorphic to a category whose objects and morphisms are well–known and understood. This leads to the concept of *presentation of a monad.* Before introducing it, we recall some basic definitions of universal algebra (see [17] for a standard introduction).

▶ **Definition 7.** *A signature Σ is a set of function symbols each having its own finite arity. We denote with $\mathcal{T}(X, \Sigma)$ the set of terms built from a set of generators X with the function symbols of Σ. An equational theory $\mathtt{Th}$ of type Σ is a set $\mathtt{Th} \subseteq \mathcal{T}(X, \Sigma) \times \mathcal{T}(X, \Sigma)$ of equations between terms $\mathcal{T}(X, \Sigma)$ closed under deducibility in the logical apparatus of equational logic. Given a set $E \subseteq \mathcal{T}(X, \Sigma) \times \mathcal{T}(X, \Sigma)$ of equations, the theory induced by E is the smallest* *equational theory containing E. The models of a theory* Th *are* Σ–algebras of the theory Th, *i.e., structures $(A, \{f^A\}_{f\in\Sigma})$ consisting of a set A and operations $f^A : A^{ar(f)} \to A$, for each operation symbol $f \in \Sigma$ having arity $ar(f)$, satisfying all (universally quantified) equations in* Th*. A homomorphism from $(A, \{f^A\}_{f\in\Sigma})$ to $(B, \{f^B\}_{f\in\Sigma})$ is a function $g : A \to B$ such that $g(f^A(a_1, \dots, a_n)) = f^B(g(a_1), \dots, g(a_n))$, for all $f \in \Sigma$. We denote with* $\mathbf{A}(\mathtt{Th})$ *the category whose objects are models of the theory* Th *and morphisms are homomorphisms.*

▶ **Definition 8** (Presentation of **Set** monads)**.** *Let $\mathcal{M}$ be a monad on* **Set***. A presentation of $\mathcal{M}$ is an equational theory* Th *such that the categories* $\mathbf{EM}(\mathcal{M})$ *and* $\mathbf{A}(\mathtt{Th})$ *are isomorphic.*

In what follows we introduce equational theories that are presentations of the three **Set** monads $\mathcal{P}$, $\mathcal{D}$ and $\mathcal{C}$ introduced earlier.

▶ **Definition 9.** *The theory* $\mathtt{Th}_{SL}$ *of semilattices is the theory having as signature $\Sigma_{SL} = \{\oplus\}$ and equations stating that $\oplus$ is associative, commutative, and idempotent:*
(A) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ *(C)* $x \oplus y = y \oplus x$ *(I)* $x \oplus x = x$.

▶ **Definition 10.** *The theory* $\mathtt{Th}_{CA}$ *of convex algebras has signature $\Sigma_{CA} = \{+_p\}_{p\in(0,1)}$ and, for all $p, q \in (0, 1)$, the equations for probabilistic associativity, commutativity, and idempotency:*
(A_p) $(x +_q y) +_p z = x +_{pq} (y +_{\frac{p(1-q)}{1-pq}} z)$ *(C_p)* $x +_p y = y +_{1-p} x$ *(I_p)* $x +_p x = x$.

▶ **Definition 11.** *The theory* $\mathtt{Th}_{CS}$ *of convex semilattices is the theory with signature $\Sigma_{CS} = (\{\oplus\} \cup \{+_p\}_{p\in(0,1)})$ where $\oplus$ satisfies the equations of semilattices, $+_p$ satisfies the equations of convex algebras for every $p \in (0, 1)$, and, furthermore, for every $p \in (0, 1)$ the following distributivity equation (D) is satisfied: $x +_p (y \oplus z) = (x +_p y) \oplus (x +_p z)$.*

The following proposition collects known results in the literature (see [48, 24, 32, 13]).

▶ **Proposition 12.**
1. *The theory* $\mathtt{Th}_{SL}$ *of semilattices is a presentation of $\mathcal{P}$, i.e.,* $\mathbf{A}(\mathtt{Th}_{SL}) \cong \mathbf{EM}(\mathcal{P})$.
2. *The theory* $\mathtt{Th}_{CA}$ *of convex algebras is a presentation of $\mathcal{D}$, i.e.,* $\mathbf{A}(\mathtt{Th}_{CA}) \cong \mathbf{EM}(\mathcal{D})$.
3. *The theory* $\mathtt{Th}_{CS}$ *of convex semilattices is a presentation of $\mathcal{C}$, i.e.,* $\mathbf{A}(\mathtt{Th}_{CS}) \cong \mathbf{EM}(\mathcal{C})$.

2.1.1 One Application: Representation of Term Algebras

Having presentations of **Set** monads as categories of algebras of equational theories is mathematically convenient for several reasons. One useful application, especially in the field of program semantics, are representation theorems for free algebras, which are, up to isomorphism, term algebras.

In this section we assume the reader to be familiar with the concept of free object in a category (see, e.g., [3, §10.3]). The free object generated by X in the category $\mathbf{EM}(\mathcal{M})$ is the $\mathcal{M}$–algebra $(\mathcal{M}(X), \mu^{\mathcal{M}}_X)$. The free object generated by X in the category $\mathbf{A}(\mathtt{Th})$ is the term algebra, i.e., the algebra whose carrier is $\mathcal{T}(X, \Sigma)_{/\mathtt{Th}}$, the set of Σ–terms constructed from the set of generators X taken modulo the equations of the theory Th, and with operations defined on equivalences classes, that is, $f([t_1]_{/\mathtt{Th}}, \dots, [t_n]_{/\mathtt{Th}}) = [f(t_1, \dots, t_n)]_{/\mathtt{Th}}$ for each $f \in \Sigma$. These characterisations, together with the fact that free objects are unique up to isomorphism, can be used to derive the following result.

▶ **Proposition 13.** *Let $\mathcal{M}$ be a monad on* **Set** *and let $F : \mathbf{A}(\mathtt{Th}) \cong \mathbf{EM}(\mathcal{M})$ be a presentation of $\mathcal{M}$ in terms of the equational theory* Th *of type Σ. Then the term algebra $\mathcal{T}(X, \Sigma)_{/\mathtt{Th}}$ and the free Eilenberg-Moore algebra $(\mathcal{M}(X), \mu^{\mathcal{M}}_X)$ are isomorphic (via F).*

In other words, a presentation theorem for $\mathcal{M}$ provides automatically representation results for term algebras via the known semantic behaviour of the multiplication of $\mathcal{M}$.

▶ **Example 14.** The presentation of the monad $\mathcal{C}$ in terms of the theory of convex semilattices implies that the free convex semilattice generated by X is isomorphic with the convex semilattice $(\mathcal{C}X, \oplus, +_p)$ where $S_1 \oplus S_2 = cc(S_1 \cup S_2)$ (convex union) and $S_1 +_p S_2 = \mathtt{WMS}(pS_1 + (1-p)S_2)$ (weighted Minkowski sum), for all $S_1, S_2 \in \mathcal{C}(X)$. In other words, the set $\mathcal{T}(X, \Sigma_{CS})_{/\mathtt{Th}_{CS}}$ of convex semilattice terms modulo the equational theory of convex semilattices can be identified with the set $\mathcal{C}(X)$ of finitely generated convex sets of finitely supported probability distributions on X. The isomorphism is explicitly given in [14] by the function $\kappa : \mathcal{C}(X) \to \mathcal{T}(X, \Sigma_{CS})_{/\mathtt{Th}_{CS}}$ defined as $\kappa(S) = [\bigoplus_{\Delta \in \mathtt{UB}(S)} (\sum\nolimits^{+}_{x \in supp(\Delta)} \Delta(x)\, x)]_{/\mathtt{Th}_{CS}}$, where $\bigoplus_{i\in I} x_i$ and $\sum\nolimits^{+}_{i \in I} p_i\, x$ are respectively notations for the binary operations $\oplus$ and $+_p$ extended to operations of arity I, for I finite (see, e.g., [47, 12]). We remark that the equation $x \oplus y = x \oplus y \oplus (x +_p y)$, which explicitly expresses closure under taking convex combinations, is derivable from the theory of convex semilattices (see, e.g., [14, Lemma 14]), and that this derivation critically uses the distributivity axiom (D).

3 Monads on Met and Quantitative Equational Theories

In Section 2 we have considered monads in the category **Set**. We now shift our focus to monads in the category **EMet** of extended metric spaces and non–expansive functions. The category **EMet** provides a natural mathematical setting for developing the semantics of programs exhibiting quantitative behaviour such as, e.g., probabilistic choice. It is indeed appropriate in this setting to replace the usual notion of program equivalence with the more informative notion of program distance (see, e.g., [45, 27, 15, 23, 16]).

▶ **Definition 15.** *An extended metric space is a pair (X, d) such that X is a set and $d : X \times X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is a function, called the* metric*, satisfying the following properties: $d(x, y) = 0$ if and only if $x = y$, $d(x, y) = d(y, x)$, and $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$. A function $f : X \to Y$ between two extended metric spaces (X, d_X) and (Y, d_Y) is called non–expansive (a.k.a. 1–Lipschitz) if $d_Y(f(x_1), f(x_2)) \leq d_X(x_1, x_2)$ for all $x_1, x_2 \in X$. We denote with* **EMet** *the category whose objects are extended metric spaces and whose morphisms are non–expansive maps.*

Since we only work with extended metric spaces, in the rest of this paper we will systematically omit the adjective "extended". Given two metrics d_1, d_2 on X, we write $d_1 \sqsubseteq d_2$ if for all $x, x' \in X$, it holds that $d_1(x, x') \leq d_2(x, x')$. Let (Y, d) be a metric space, X a set and $f : X \to Y$. We write $d\langle f, f\rangle$ for the metric on X defined as $d\langle f, f\rangle(x_1, x_2) = d(f(x_1), f(x_2))$. Let $d_{\mathbb{R}}$ be the Euclidean metric on $\mathbb{R}$ defined as $d_{\mathbb{R}}(r_1, r_2) = |r_1 - r_2|$. If (X, d) is a metric space, we simply say that $f : X \to [0, 1]$ is non–expansive to mean that $f : (X, d) \to ([0, 1], d_{\mathbb{R}})$ is non–expansive. The metric d of a metric space (X, d) induces a topology on X whose open sets are generated by the *open balls* of the form $B(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$, for $x \in X$ and $\epsilon > 0$. A subset $Y \subseteq X$ is called *compact* if each of its open covers has a finite subcover. Every compact set Y is closed and bounded (i.e., the distance between elements in Y is bounded by some real number). The collection of non–empty compact subsets of a metric space (X, d) is denoted by $\mathtt{Comp}(X, d)$. Note that every finite subset of X belongs to $\mathtt{Comp}(X, d)$.

The **Set** monads $\mathcal{P}$ and $\mathcal{D}$ defined in Section 2 can be extended to monads in **EMet**. These extensions are well–known and are based on metric liftings constructions due to Hausdorff and Kantorovich (see [34] for a standard reference).

▶ **Definition 16** (Hausdorff Lifting). *Let (X, d) be a metric space. The Hausdorff lifting of d is a metric $H(d)$ on $\texttt{Comp}(X, d)$, the collection of non–empty compact subsets of X, defined as follows for any pair $X_1, X_2 \in \texttt{Comp}(X, d)$:*

$$H(d)\big(X_1, X_2\big) = \max\big\{ \sup_{x_1\in X_1} \inf_{x_2\in X_2} d(x_1, x_2) \quad , \quad \sup_{x_2\in X_2} \inf_{x_1\in X_1} d(x_1, x_2)\big\}.$$

This leads to the well–known *hyperspace* monad $\mathcal{V}$ on **EMet** ([31], see also [34]).[3]

▶ **Definition 17.** *The* hyperspace *monad $(\mathcal{V}, \eta^{\mathcal{V}}, \mu^{\mathcal{V}})$ on* **EMet** *is defined as follows. Given an object (X, d) in* **EMet***, $\mathcal{V}(X, d) = \big(\texttt{Comp}(X, d), H(d)\big)$, the metric space of non–empty compact subsets of X equipped with the Hausdorff distance. Given a non–expansive map $f : (X, d_X) \to (Y, d_Y)$, $\mathcal{V}(f)(X') = \bigcup_{x\in X'} f(x)$. The unit $\eta^{\mathcal{V}}_{(X,d)} : (X, d) \to \mathcal{V}(X, d)$ is defined as $\eta^{\mathcal{V}}_{(X,d)}(x) = \{x\}$, and the multiplication $\mu^{\mathcal{V}}_{(X,d)} : \mathcal{V}\mathcal{V}(X, d) \to \mathcal{V}(X, d)$ is defined as $\mu^{\mathcal{V}}_{(X,d)}(\{X_i\}_{i\in I}) = \bigcup_i X_i$.*

The restriction of the monad $\mathcal{V}$ to finite (hence compact) subsets leads to the non–empty finite powerset monad on **EMet**, which we denote with $\hat{\mathcal{P}}$ to distinguish it from the **Set** monad $\mathcal{P}$.

▶ **Definition 18.** *The* non–empty finite powerset *monad $(\hat{\mathcal{P}}, \eta^{\hat{\mathcal{P}}}, \mu^{\hat{\mathcal{P}}})$ on* **EMet** *is defined as follows. Given an object (X, d) in* **EMet***, $\hat{\mathcal{P}}(X, d) = \big(\mathcal{P}(X), H(d)\big)$, the collection of finite non–empty subsets of X equipped with the Hausdorff distance. The action of $\hat{\mathcal{P}}$ on morphisms, the unit $\eta^{\hat{\mathcal{P}}}$ and the multiplication $\mu^{\hat{\mathcal{P}}}$ are defined as for the* **Set** *monad $\mathcal{P}$ (or, equivalently, as for the $\mathcal{V}$ monad on* **EMet** *restricted to finite sets).*

Next, we introduce the Kantorovich lifting on finitely supported distributions [34].

▶ **Definition 19** (Kantorovich Lifting). *Let (X, d) be a metric space. The Kantorovich lifting of d is a metric $K(d)$ on $\mathcal{D}(X)$, the collection of finitely supported probability distributions on X, defined as follows for any pair $\Delta_1, \Delta_2 \in \mathcal{D}(X)$:*

$$K(d)(\Delta_1, \Delta_2) = \inf_{\omega\in Coup(\Delta_1,\Delta_2)} \Big(\sum_{(x_1,x_2)\in X\times X} \omega(x_1, x_2) \cdot d(x_1, x_2)\Big)$$

where $Coup(\Delta_1, \Delta_2)$ is defined as the collection of couplings of Δ_1 and Δ_2, i.e., the collection of probability distributions on the product space $X \times X$ such that the marginals of ω are Δ_1 and Δ_2. Formally, $Coup(\Delta_1, \Delta_2) = \{\omega \in \mathcal{D}(X \times X) \mid \mathcal{D}(\pi_1)(\omega) = \Delta_1 \text{ and } \mathcal{D}(\pi_2)(\omega) = \Delta_2\}$ where $\pi_1 : X_1 \times X_2 \to X_1$ and $\pi_2 : X_1 \times X_2 \to X_2$ are the projection functions.

We can now introduce the following version of the finitely supported probability distribution monad on **EMet**, which we denote with $\hat{\mathcal{D}}$ to distinguish it from the **Set** monad $\mathcal{D}$.

▶ **Definition 20.** *The* finitely supported probability distribution *monad $(\hat{\mathcal{D}}, \eta^{\hat{\mathcal{D}}}, \mu^{\hat{\mathcal{D}}})$ on* **EMet** *is defined as follows. Given an object (X, d) in* **EMet***, $\hat{\mathcal{D}}(X, d) = \big(\mathcal{D}(X), K(d)\big)$, the collection of finitely supported probability distributions on X equipped with the Kantorovich distance. The action of $\hat{\mathcal{D}}$ on morphisms, the unit $\eta^{\hat{\mathcal{D}}}$, and the multiplication $\mu^{\hat{\mathcal{D}}}$ are defined as for the* **Set** *monad $\mathcal{D}$.*

The fact that the above definitions are correct (i.e., that $\hat{\mathcal{D}}$ is a functor and that $\eta^{\hat{\mathcal{D}}}$ and $\mu^{\hat{\mathcal{D}}}$ are non–expansive and satisfy the monad laws) is well–known (see, e.g., [34, 15, 8]).

[3] This monad, defined on the category **Met** of ordinary (i.e., non–extended) metric spaces, is essentially due to Hausdorff [31]. See, e.g., [34] for a detailed exposition.

3.1 Quantitative Equational Theories and Quantitative Algebras

We provide here the essential definitions and results of the framework developed by Mardare, Panangaden, and Plotkin in [36] (see also [7, 37, 5, 38]). In what follows, a signature Σ is fixed. Recall that $\mathcal{T}(X,\Sigma)$ denotes the set of terms constructed from X using the function symbols in Σ. A substitution is a map of type $\sigma : X \to \mathcal{T}(X,\Sigma)$. As usual, to any interpretation $\iota : X \to A$ of the variables into a set corresponds, by homomorphic extension, a unique map $\iota : \mathcal{T}(X,\Sigma) \to A$.

▶ **Definition 21** (Quantitative Equational Theory). *A* quantitative equation *is an expression of the form $t =_\epsilon s$, where $t, s \in \mathcal{T}(X,\Sigma)$ and $\epsilon \in \mathbb{R}_{\geq 0}$. We denote with $E(\Sigma)$ the collection of all quantitative equations. We use the letters Γ, Θ to range over subsets of $E(\Sigma)$. A* quantitative inference *is an element of $2^{E(\Sigma)} \times E(\Sigma)$, i.e., a pair $(\Gamma, t =_\epsilon s)$ where $\Gamma \subseteq E(\Sigma)$ and $t =_\epsilon s$ is a quantitative equation. Note that Γ needs not be finite. A* deducibility relation *is a set of quantitative inferences $\vdash \subseteq 2^{E(\Sigma)} \times E(\Sigma)$ closed under the following conditions which are stated for arbitrary $s, t, u \in \mathcal{T}(X,\Sigma)$, $\epsilon, \epsilon' \in \mathbb{R}_{\geq 0}$, $\Gamma, \Theta \subseteq E(\Sigma)$, and $f \in \Sigma$: (Notation: we use the infix notation $\Gamma \vdash t =_\epsilon s$ to mean that $(\Gamma, t =_\epsilon s) \in \vdash$)*

(Refl) $\emptyset \vdash t =_0 t$ *(Symm)* $\{t =_\epsilon s\} \vdash s =_\epsilon t$ *(Triang)* $\{t =_\epsilon u, u =_{\epsilon'} s\} \vdash t =_{\epsilon+\epsilon'} s$

(Max) $\{t =_\epsilon s\} \vdash t =_{\epsilon'} s$, *where* $\epsilon' > \epsilon$ *(Arch)* $\{t =_{\epsilon'} s\}_{\epsilon' > \epsilon} \vdash t =_\epsilon s$

(NExp) $\{t_i =_\epsilon s_i\}_{i \in 1 \ldots ar(f)} \vdash f(t_1, \ldots, t_n) =_\epsilon f(s_1, \ldots s_n)$

(Subst) *if* $\Gamma \vdash t =_\epsilon s$ *then* $\{\sigma(t) =_\epsilon \sigma(s) \mid (t =_\epsilon s) \in \Gamma\} \vdash \sigma(t) =_\epsilon \sigma(s)$, *for all substitutions* σ

(Cut) *if* $\Gamma \vdash \Theta$ *and* $\Theta \vdash t =_\epsilon s$ *then* $\Gamma \vdash t =_\epsilon s$

(Assum) *if* $t =_\epsilon s \in \Gamma$ *then* $\Gamma \vdash t =_\epsilon s$, *for all* Γ, t, s, ϵ

where in (Cut) the expression $\Gamma \vdash \Theta$ means that for all $(t =_\epsilon s) \in \Theta$ it holds that $\Gamma \vdash t =_\epsilon s$. Given a set of quantitative inferences $\mathcal{U} \subseteq 2^{E(\Sigma)} \times E(\Sigma)$, the quantitative equational theory induced by $\mathcal{U}$ is the smallest deducibility relation which includes $\mathcal{U}$.

The models of quantitative theories are quantitative algebras, which we now introduce.

▶ **Definition 22** (Quantitative Algebra). *A* quantitative algebra *of type Σ is a structure $\mathbb{A} = \big(A, \{f^A\}_{f \in \Sigma}, d_A\big)$ where (A, d_A) is an extended metric space and, for each $f \in \Sigma$, the function $f^A : A^{ar(f)} \to A$ is a non–expansive map, with $A^{ar(f)}$ endowed with the* sup*–metric defined as $d_{\sup}(\{a_i\}_{i \in ar(f)}, \{b_i\}_{i \in ar(f)}) = \max_{i \in ar(f)}(d(a_i, b_i))$. A homomorphism between quantitative algebras $\mathbb{A}$ and $\mathbb{B}$ of type Σ is a non–expansive function $g : (A, d_A) \to (B, d_B)$ which preserves all operations in Σ, i.e., $g(f^A(x_1, \ldots, x_n)) = f^B(g(x_1), \ldots, g(x_n))$, for all $x_i \in A$. We say that $\mathbb{A}$ satisfies a quantitative inference $(\{s_i =_{\epsilon_i} t_i\}_{i \in I}, s =_\epsilon t)$, written $\{s_i =_{\epsilon_i} t_i\} \models_{\mathbb{A}} s =_\epsilon t$, if for every interpretation $\iota : X \to A$ of the variables X into elements of A the following holds: if for all $i \in I$, $d_A\big(\iota(s_i), \iota(t_i)\big) \leq \epsilon_i$, then $d_A\big(\iota(s), \iota(t)\big) \leq \epsilon$. We say that $\mathbb{A}$ is a model of a quantitative theory* `QTh` *if $\mathbb{A}$ satisfies every quantitative inference in* `QTh`*. We denote with $\mathbf{QA}$(*`QTh`*) the category having as objects the quantitative algebras that are models of* `QTh`*, and as arrows the non–expansive homomorphisms between quantitative algebras of type Σ.*

Every quantitative algebra of type Σ satisfies the quantitative inferences generating the deducibility relation $\vdash$ in Definition 21. We refer to [36] for proofs that all the above definitions are indeed well–defined. Two interesting quantitative theories studied in [36] are the following.[4]

[4] We remark that, in [36], the quantitative theory of convex algebras is referred to as the quantitative theory of interpolative barycentric algebras.

▶ **Definition 23** (Quantitative Semilattices). *The quantitative theory of* quantitative semilattices*, denoted by* $\mathtt{QTh}_{SL}$*, has type* Σ_{SL} *(see Definition 9) and is induced by the following quantitative inferences, for all* $\epsilon_1, \epsilon_2 \in \mathbb{R}_{\geq 0}$*:*

(A) $\emptyset \vdash x \oplus (y \oplus z) =_0 (x \oplus y) \oplus z$ *(C)* $\emptyset \vdash x \oplus y =_0 y \oplus x$ *(I)* $\emptyset \vdash x \oplus x =_0 x$

(H) $\{x_1 =_{\epsilon_1} y_1, x_2 =_{\epsilon_2} y_2\} \vdash x_1 \oplus x_2 =_{\max(\epsilon_1,\epsilon_2)} y_1 \oplus y_2$.

▶ **Definition 24** (Quantitative Convex Algebras). *The quantitative theory of* quantitative convex algebras*, denoted by* $\mathtt{QTh}_{CA}$*, has type* Σ_{CA} *(see Definition 10) and is induced by the following quantitative inferences, for all* $p, q \in (0,1)$ *and* $\epsilon_1, \epsilon_2 \in \mathbb{R}_{\geq 0}$*:*

(A_p*)* $\emptyset \vdash (x +_q y) +_p z =_0 x +_{pq} (y +_{\frac{p(1-q)}{1-pq}} z)$ *(*C_p*)* $\emptyset \vdash x +_p y =_0 y +_{1-p} x$

(I_p*)* $\emptyset \vdash x +_p x =_0 x$ *(K)* $\{x_1 =_{\epsilon_1} y_1, x_2 =_{\epsilon_2} y_2\} \vdash x_1 +_p x_2 =_{p\cdot\epsilon_1+(1-p)\cdot\epsilon_2} y_1 +_p y_2$.

In other words, the theories $\mathtt{QTh}_{SL}$ and $\mathtt{QTh}_{CA}$ are obtained by taking the equational axioms of semilattices and convex algebras respectively (Definitions 9 and 10), replacing the equality $(=)$ with $(=_0)$, and by introducing the quantitative inferences (H) and (K) respectively.

A general result from [36, §5] states that free objects always exist in $\mathbf{QA}(\mathtt{QTh})$, for any $\mathtt{QTh}$, and they are isomorphic to term quantitative algebras for $\mathtt{QTh}$. Moreover, such free objects are concretely identified for two relevant theories:

▶ **Theorem 25** ([36, Cor 9.4 and 10.6]). *The following hold:*

- *The free quantitative semilattice in* $\mathbf{QA}(\mathtt{QTh}_{SL})$ *generated by a metric space* (X,d) *is isomorphic to the metric space* $\hat{\mathcal{P}}(X,d) = \big(\mathcal{P}(X), H(d)\big)$.
- *The free quantitative convex algebra in* $\mathbf{QA}(\mathtt{QTh}_{CA})$ *generated by a metric space* (X,d) *is isomorphic to the metric space* $\hat{\mathcal{D}}(X,d) = \big(\mathcal{D}(X), K(d)\big)$.

We remark that the above theorem from [36] falls short from a full presentation result stating the isomorphisms of categories $\mathbf{QA}(\mathtt{QTh}_{SL}) \cong \mathbf{EM}(\hat{\mathcal{P}})$ and $\mathbf{QA}(\mathtt{QTh}_{CA}) \cong \mathbf{EM}(\hat{\mathcal{D}})$. This latter more general statement does indeed hold and can be obtained, with some minor extra work, from the technical machinery developed in [36] (see Footnote 2).

4 The Monad $\hat{\mathcal{C}}$ on the Category of Metric Spaces

In this section we introduce a **EMet** version of the **Set** monad $\mathcal{C}$, and we denote it with $\hat{\mathcal{C}}$. The monad $\hat{\mathcal{C}}$ is obtained by composing the Hausdorff lifting H and the Kantorovich lifting K introduced in the previous section.

▶ **Proposition 26.** *Let* (X,d) *be a metric space and let* $S \in \mathtt{Comp}(\mathcal{D}(X), K(d))$*. Then* $cc(S) \in \mathtt{Comp}(\mathcal{D}(X), K(d))$*, i.e., the convex closure of* S *is also compact.*

▶ **Corollary 27.** *Let* (X,d) *be a metric space. If* $S \in \mathcal{C}(X)$ *then* $S \in \mathtt{Comp}(\mathcal{D}(X), K(d))$*.*

Corollary 27 implies that, given a metric space (X,d), the collection $\mathcal{C}(X)$ of finitely generated non–empty convex sets of distributions on X can be endowed with the subspace metric of $\mathcal{V}(\hat{\mathcal{D}}(X,d))$, and therefore $(\mathcal{C}(X), HK(d))$ is a metric space, with $HK(d) = H(K(d))$. This observation leads to the following definition.

▶ **Definition 28** (Monad $\hat{\mathcal{C}}$). *The* finitely generated non–empty convex powerset of finitely supported probability distributions *monad* $(\hat{\mathcal{C}}, \eta^{\hat{\mathcal{C}}}, \mu^{\hat{\mathcal{C}}})$ *on* **EMet** *is defined as follows. Given an object* (X,d) *in* **EMet***,* $\hat{\mathcal{C}}(X,d) = \big(\mathcal{C}(X), HK(d)\big)$*. The action of* $\hat{\mathcal{C}}$ *on morphisms, the monad unit* $\eta^{\hat{\mathcal{C}}}$*, and the monad multiplication* $\mu^{\hat{\mathcal{C}}}$ *are defined as for the* **Set** *monad* $\mathcal{C}$ *(Definition 5).*

The rest of this section is devoted to the proof that the above definition is well–specified, i.e., that $\hat{\mathcal{C}}$ is indeed a monad on **EMet**. First, one needs to verify that $\hat{\mathcal{C}}$ is a functor on **EMet**. This follows immediately from the definition, Corollary 27, and $\mathcal{C}$ being a functor on **Set**. It then remains to verify that the unit $\eta^{\hat{\mathcal{C}}}$ and the multiplication $\mu^{\hat{\mathcal{C}}}$ of $\hat{\mathcal{C}}$ are indeed morphisms in **EMet** (i.e., they are non-expansive functions) and that they satisfy the monad laws of Definition 1. The fact that the laws are satisfied follows directly from the definitions $\mu^{\hat{\mathcal{C}}} = \mu^{\mathcal{C}}$ and $\eta^{\hat{\mathcal{C}}} = \eta^{\mathcal{C}}$ and the fact that $\mathcal{C}$ is a monad on **Set** (hence $\mu^{\mathcal{C}}$ and $\eta^{\mathcal{C}}$ satisfy the monad laws). Then it only remains to verify that $\eta^{\hat{\mathcal{C}}}$ and $\mu^{\hat{\mathcal{C}}}$ are non–expansive. It is straightforward to verify that $\eta^{\hat{\mathcal{C}}}$ is an isometric (hence non–expansive) embedding of (X, d) into $\big(\mathcal{C}(X), HK(d)\big)$. Proving that $\mu^{\hat{\mathcal{C}}}$ is non–expansive, instead, does not seem straightforward and requires some detailed calculations. We state this result as a theorem.

▶ **Theorem 29.** *Let (X, d) be a metric space in* **EMet**. *Then $\mu^{\hat{\mathcal{C}}}_{(X,d)} : \hat{\mathcal{C}}\hat{\mathcal{C}}(X, d) \to \hat{\mathcal{C}}(X, d)$ is a non–expansive function, i,e., using functional notation, $HK(d)\langle\mu^{\hat{\mathcal{C}}}, \mu^{\hat{\mathcal{C}}}\rangle \sqsubseteq HKHK(d)$.*

4.1 Sketch of the Proof of Theorem 29

The key result to prove is Lemma 32, stating that the weighted Minkowski sum function `WMS` is non–expansive. This is obtained by exploiting a key property of the HK metric (see Lemma 31) called convexity. It might well be that both these results have already appeared in the literature in some form or another or are known as folklore by specialists. We present here a direct proof.

▶ **Definition 30** (Convex metric). *Let $(X, \{+_p\}_{p\in(0,1)})$ be a convex algebra, i.e., a set X equipped with operations $+_p : X \times X \to X$ satisfying the axioms of Definition 10. Let $d : X \times X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ be a metric on X. We say that d is convex if $d(x_1 +_p x_2, y_1 +_p y_2) \leq d(x_1, y_1) +_p d(x_2, y_2)$ holds for all $x_1, x_2, y_1, y_2 \in X$ and $p \in (0, 1)$, where $d(x_1, y_1) +_p d(x_2, y_2) = p \cdot d(x_1, y_1) + (1 - p) \cdot d(x_2, y_2)$ with the convention that $\infty +_p x = x +_q \infty = \infty +_r \infty = \infty$ for all $p, q, r \in (0, 1)$ and $x \in X$.*

It is well known that the Kantorovich metric $K(d)$ is convex. The following lemma states that also the Hausdorff–Kantorovich metric $HK(d)$, defined on the collection $\mathcal{C}(X)$ of non–empty finitely generated convex sets of distributions, which carries the structure of a convex semilattice (see Example 14) and thus also of a convex algebra, is convex.

▶ **Lemma 31.** *Let (X, d) be a metric space. The metric $HK(d)$ on the convex algebra $(\mathcal{C}(X), \{+_p\}_{p\in(0,1)})$, with $S_1 +_p S_2 = \mathtt{WMS}(pS_1 + (1 - p)S_2)$, is convex.*

Using the convexity of HK it is possible to prove that the `WMS` function is non–expansive.

▶ **Lemma 32.** *Let (X, d) be a metric space. The function $\mathtt{WMS} : \hat{\mathcal{D}}(\hat{\mathcal{C}}(X, d)) \to \hat{\mathcal{C}}(X, d)$ (see Definition 5) is non–expansive, i.e. $HK(d)\langle\mathtt{WMS}, \mathtt{WMS}\rangle \sqsubseteq KHK(d)$.*

Lastly, we state the following two useful properties of the Hausdorff lifting.

▶ **Proposition 33.** *Let d, d' be two metrics over X such that $d \sqsubseteq d'$. Then $H(d) \sqsubseteq H(d')$.*

▶ **Proposition 34.** *Let (X, d_X) and (Y, d_Y) be metric spaces, let $f : X \to Y$ with $d_X = d_Y\langle f, f\rangle$ (i.e., $d_X(x_1, x_2) = d_Y(f(x_1), f(x_2))$. Then $H(d_X) = H(d_Y)\langle\mathcal{V}(f), \mathcal{V}(f)\rangle$.*

Proof of Theorem 29. We need to show that $HK(d)\langle\mu^{\hat{\mathcal{C}}}, \mu^{\hat{\mathcal{C}}}\rangle \sqsubseteq HKHK(d)$.

Since $\mathcal{V}$ is a monad on **EMet** (Definition 17), $\mu^{\mathcal{V}}$ is non-expansive, i.e., $H(d)\langle\mu^{\mathcal{V}}, \mu^{\mathcal{V}}\rangle \sqsubseteq HH(d)$. By applying this to the metric $K(d)$, we derive

$$HK(d)\langle\mu^{\mathcal{V}}, \mu^{\mathcal{V}}\rangle \sqsubseteq HHK(d). \tag{1}$$

By definition $\mu^{\hat{\mathcal{C}}} = \mu^{\mathcal{V}} \circ \mathcal{V}(\mathtt{WMS})$ (i.e., $S \mapsto \bigcup\{\mathtt{WMS}(\Delta) \mid \Delta \in S\}$) and therefore:

$$\begin{aligned} HK(d)\langle \mu^{\hat{\mathcal{C}}}, \mu^{\hat{\mathcal{C}}} \rangle &= HK(d)\langle \mu^{\mathcal{V}} \circ \mathcal{V}(\mathtt{WMS}), \mu^{\mathcal{V}} \circ \mathcal{V}(\mathtt{WMS}) \rangle \\ &= HK(d)\langle \mu^{\mathcal{V}}, \mu^{\mathcal{V}} \rangle \langle \mathcal{V}(\mathtt{WMS}), \mathcal{V}(\mathtt{WMS}) \rangle \end{aligned}$$

Thus, by (1) we can derive

$$HK(d)\langle \mu^{\hat{\mathcal{C}}}, \mu^{\hat{\mathcal{C}}} \rangle \sqsubseteq HHK(d)\langle \mathcal{V}(\mathtt{WMS}), \mathcal{V}(\mathtt{WMS}) \rangle. \tag{2}$$

Moreover, by the non-expansiveness of WMS (Lemma 32), we know that

$$HK(d)\langle \mathtt{WMS}, \mathtt{WMS} \rangle \sqsubseteq KHK(d)$$

which implies by the monotonicity of H (Proposition 33) that

$$H(HK(d)\langle \mathtt{WMS}, \mathtt{WMS} \rangle) \sqsubseteq HKHK(d). \tag{3}$$

By Proposition 34, we can rewrite the left-hand term of (3) as follows

$$H(HK(d)\langle \mathtt{WMS}, \mathtt{WMS} \rangle) = HHK(d)\langle \mathcal{V}(\mathtt{WMS}), \mathcal{V}(\mathtt{WMS}) \rangle$$

and thus we derive from (3):

$$HHK(d)\langle \mathcal{V}(\mathtt{WMS}), \mathcal{V}(\mathtt{WMS}) \rangle \sqsubseteq HKHK(d). \tag{4}$$

Lastly, by (2) and (4): $HK(d)\langle \mu^{\hat{\mathcal{C}}}, \mu^{\hat{\mathcal{C}}} \rangle \sqsubseteq HHK(d)\langle \mathcal{V}(\mathtt{WMS}), \mathcal{V}(\mathtt{WMS}) \rangle \sqsubseteq HKHK(d)$. ◀

5 Presentation of the Monad $\hat{\mathcal{C}}$

In this section we present the main result of this work and show that the monad $\hat{\mathcal{C}}$ on **EMet**, introduced in Section 4, is presented by quantitative convex semilattices.

▶ **Definition 35.** *The quantitative equational theory of* quantitative convex semilattices*, denoted by* $\mathtt{QTh}_{CS}$*, is the quantitative theory over the signature* $\Sigma_{CS} = (\{\oplus\} \cup \{+_p\}_{p\in(0,1)})$ *of convex semilattices induced by the following set of quantitative inferences:*

- *the quantitative inferences (A), (C), (I) and (H) inducing the quantitative theory of semilattices (see Definition 23),*
- *the quantitative inferences* (A_p), (C_p), (I_p)*, and (K) inducing the quantitative theory of convex algebras (see Definition 24),*
- *for every* $p \in (0,1)$*, the quantitative inference (D)* $\emptyset \vdash x +_p (y \oplus z) =_0 (x +_p y) \oplus (x +_p z)$*.*

The following is the main result of this work.

▶ **Theorem 36.** *The quantitative equational theory* $\mathtt{QTh}_{CS}$ *of quantitative convex semilattices is a presentation of the monad* $\hat{\mathcal{C}}$*, that is,* $\mathbf{QA}(\mathtt{QTh}_{CS}) \cong \mathbf{EM}(\hat{\mathcal{C}})$*.*

As one direct corollary of this general statement we automatically get the following result (cf. with Theorem 25) characterising free quantitative convex semilattices, which, by [36, §5], are in turn isomorphic to term quantitative algebras for $\mathtt{QTh}_{CS}$.

▶ **Corollary 37.** *The free quantitative algebra in* $\mathbf{QA}(\mathtt{QTh}_{CS})$ *generated by a metric space* (X, d) *is isomorphic to* $\hat{\mathcal{C}}(X, d)$*, the metric space of finitely generated convex sets of probability distributions metrized by the Hausdorff–Kantorovich metric* $HK(d)$*.*

We prove Theorem 36 by explicitly defining a pair of functors $\mathcal{F} : \mathbf{EM}(\hat{\mathcal{C}}) \to \mathbf{QA}(\mathtt{QTh}_{CS})$ and $\mathcal{G} : \mathbf{QA}(\mathtt{QTh}_{CS}) \to \mathbf{EM}(\hat{\mathcal{C}})$ and proving that they are isomorphisms of categories, i.e., that $\mathcal{G} \circ \mathcal{F} = id_{\mathbf{EM}(\hat{\mathcal{C}})}$ and $\mathcal{F} \circ \mathcal{G} = id_{\mathbf{QA}(\mathtt{QTh}_{CS})}$. In the following sections, we exhibit such functors and show that they are well-defined isomorphisms.

▶ Remark 38. A recent result (Theorem 4.2 of [7]), showing that, for any quantitative equational theory, the category of Eilenberg-Moore algebras of the term monad and the category $\mathbf{QA}(\mathtt{QTh}_{CS})$ are isomorphic, might provide an alternative route to obtain the result of Theorem 36. Our proof technique has the virtue of concretely exhibiting the functors witnessing the isomorphism.

5.1 The functor $\mathcal{F} : \mathrm{EM}(\hat{\mathcal{C}}) \to \mathrm{QA}(\mathtt{QTh}_{CS})$

Recall from Definition 6 that an object in $\mathbf{EM}(\hat{\mathcal{C}})$ is a structure $((X, d), \alpha)$ where (X, d) is a metric space and $\alpha : (\mathcal{C}(X), HK(d)) \to (X, d)$ is a non-expansive function satisfying $\alpha \circ \eta^{\hat{\mathcal{C}}}_X = id_X$ and $\alpha \circ \hat{\mathcal{C}}\alpha = \alpha \circ \mu^{\hat{\mathcal{C}}}_X$. A morphism $f : ((X, d_X), \alpha_X) \to ((Y, d_Y), \alpha_Y)$ in $\mathbf{EM}(\hat{\mathcal{C}})$ is a non–expansive function $f : X \to Y$ such that $f \circ \alpha_X = \alpha_Y \circ \hat{\mathcal{C}}(f)$.

▶ **Definition 39** (Functor $\mathcal{F}$). *We define $\mathcal{F} : \mathbf{EM}(\hat{\mathcal{C}}) \to \mathbf{QA}(\mathtt{QTh}_{CS})$ as follows:*

- *on objects: $\mathcal{F}((X, d), \alpha) = (X, \Sigma^{\alpha}_{CS}, d)$*
  with $\Sigma^{\alpha}_{CS} = (\{\oplus^{\alpha}\} \cup \{+^{\alpha}_p\}_{p\in(0,1)})$ the interpretation of the convex semilattice operations $\oplus$ and $+_p$ as $x_1 \oplus^{\alpha} x_2 = \alpha(cc\{\delta(x_1), \delta(x_2)\})$ and $x_1 +^{\alpha}_p x_2 = \alpha(\{px_1 + (1 - p)x_2\})$,
- *on morphisms: $\mathcal{F}(f) = f$, with $f : X \to Y$ seen as a non–expansive map from X to Y.*

We now prove that the functor $\mathcal{F}$ is well-defined. First, on objects, we need to show that $\mathcal{F}((X, d), \alpha)$ is indeed a quantitative algebra satisfying the quantitative inferences of the theory $\mathtt{QTh}_{CS}$. To show that $(X, \Sigma^{\alpha}_{CS}, d)$ is a quantitative algebra (Definition 22), since (X, d) is a metric space, we only need to verify that the operations $\oplus^{\alpha}$ and $+^{\alpha}_p$ are non–expansive.

▶ **Lemma 40.** *The operations $\oplus^{\alpha}$ and $+^{\alpha}_p$, for all $p \in (0, 1)$, are non–expansive.*

Proof. Using functional notation we have $\oplus^{\alpha} = \alpha \circ cc \circ \mathcal{P}\eta^{\mathcal{D}}_X \circ (\lambda x_1, x_2.\{x_1, x_2\})$. The function α is non–expansive by assumption. $\mathcal{P}\eta^{\mathcal{D}}_X$ is non-expansive by $\hat{\mathcal{P}}$ and $\hat{\mathcal{D}}$ being monads on $\mathbf{EMet}$. The functions $\lambda x_1, x_2.\{x_1, x_2\} : (X, d) \times (X, d) \to \hat{\mathcal{P}}(X, d)$ and $cc : \hat{\mathcal{P}}\hat{\mathcal{D}}(X, d) \to \hat{\mathcal{C}}(X, d)$ are non–expansive as well. Hence $\oplus^{\alpha}$ is non–expansive as composition of non–expansive maps. Similarly, we have $+^{\alpha}_p = \alpha \circ \eta^{\mathcal{P}}_{\mathcal{D}(X)} \circ \big(\lambda x_1, x_2.(px_1 + (1 - p)x_2)\big)$ and all operations involved are non–expansive. ◀

As $\mathcal{F}((X, d), \alpha)$ is a quantitative algebra, it satisfies all the quantitative inferences of Definition 21. It only remains to show that the quantitative inferences of the theory $\mathtt{QTh}_{CS}$ (Definition 35) are also satisfied. For each of the quantitative inferences (A, C, I, A_p, C_p, I_p, D), which are of the form $\emptyset \vdash s =_0 t$, we need to show that the equality $s = t$ holds (universally quantified) in $(X, \Sigma^{\alpha}_{CS}, d)$. This amounts to showing that the algebra $(X, \Sigma^{\alpha}_{CS})$ (with the metric d forgotten) is a model of the equational theory of convex semilattices (Definition 11). This proof has no specific metric–theoretic content and is omitted here. Thus, it only remains to show that the quantitative inferences (H) and (K) are satisfied.

▶ **Lemma 41** (H). $\big\{x_1 =_{\epsilon_1} y_1, x_2 =_{\epsilon_2} y_2\big\} \models_{\mathcal{F}((X,d),\alpha)} x_1 \oplus x_2 =_{\max(\epsilon_1,\epsilon_2)} y_1 \oplus y_2$.

Proof. The quantitative inference (H) is equivalent (i.e., mutually derivable in presence of the others deductive rules of Definition 21) with the (NExp) deductive rule. This means that (H) holds in $\mathcal{F}((X, d), \alpha)$ because the operation $\oplus^{\alpha}$ is non–expansive (Lemma 40). ◀

▶ **Lemma 42** (K). $x_1 =_{\epsilon_1} y_1, x_2 =_{\epsilon_2} y_2 \models_{\mathcal{F}((X,d),\alpha)} x_1 +_p x_2 =_{p\cdot\epsilon_1+(1-p)\cdot\epsilon_2} y_1 +_p y_2$.

Proof. For arbitrary $x_1, x_2, y_1, y_2 \in X$, assume $d(x_1, y_1) \leq \epsilon_1$ and $d(x_2, y_2) \leq \epsilon_2$. Then

$$\begin{aligned} d(x_1 +^\alpha_p x_2, y_1 +^\alpha_p y_2) &= d(\alpha(\{px_1 + (1-p)x_2\}), \alpha(\{py_1 + (1-p)y_2\}) \\ &\leq HK(d)(\{px_1 + (1-p)x_2\}, \{py_1 + (1-p)y_2\}) \qquad (\alpha \text{ non-exp.}) \\ &= K(d)(px_1 + (1-p)x_2, py_1 + (1-p)y_2) \\ &\leq p \cdot d(x_1, y_1) + (1-p) \cdot d(x_2, y_2) \qquad (\text{the metric } K(d) \text{ is convex}) \\ &\leq p \cdot \epsilon_1 + (1-p) \cdot \epsilon_2 \end{aligned}$$ ◀

Hence $\mathcal{F}$ is well–defined on objects. It remains to verify that $\mathcal{F}$ is well defined on morphisms. Let $f : ((X, d), \alpha) \to ((Y, d'), \beta)$ be a morphism in $\mathbf{EM}(\hat{\mathcal{C}})$. We need to verify that $\mathcal{F}(f)$ is a morphisms in $\mathbf{QA}(\mathtt{QTh}_{CS})$, i.e., a non–expansive homomorphism of convex semilattices (see Definition 22). Since by definition $\mathcal{F}(f) = f$, the function $\mathcal{F}(f)$ is non–expansive. It remains to verify that it is a homomorphism. This proof has no specific metric–theoretic content and we omit it here.

5.2 The functor $\mathcal{G} : \mathbf{QA}(\mathtt{QTh}_{CS}) \to \mathbf{EM}(\hat{\mathcal{C}})$

Recall that an object in $\mathbf{QA}(\mathtt{QTh}_{CS})$ is a quantitative convex semilattice $\mathbb{A} = (X, \Sigma^{\mathbb{A}}_{CS}, d)$, with $\Sigma^{\mathbb{A}}_{CS} = (\{\oplus^{\mathbb{A}}\} \cup \{+^{\mathbb{A}}_p\}_{p\in(0,1)})$. Also, recall from Example 14 that there is an isomorphism κ mapping elements of $\mathcal{C}(X)$ to equivalence classes of convex semilattice terms in $\mathcal{T}(X, \Sigma_{CS})_{/\mathtt{Th}_{CS}}$. Let us define $\nu : \mathcal{C}(X) \to \mathcal{T}(X, \Sigma_{CS})$ as a choice function, mapping each $S \in \mathcal{C}(X)$ to one representative of the equivalence class $\kappa(S)$. This allows us to uniquely write down each $S \in \mathcal{C}(X)$ as a convex semilattice term:

$$\nu(S) = \bigoplus_{\Delta\in\mathtt{UB}(S)} \left(\sum_{x\in supp(\Delta)} \Delta(x)\, x \right).$$

With abuse of notation, we have used the letter X to range both over a set of variables and the carrier of $\mathbb{A}$. By interpreting each variable x with the corresponding element $x \in X$ of $\mathbb{A}$, and by homomorphic extension, we get that each term $t \in \mathcal{T}(X, \Sigma_{CS})$ can be interpreted as an element $t^{\mathbb{A}}$ of $\mathbb{A}$, and in particular $(\nu(S))^{\mathbb{A}}$ denotes an element of $\mathbb{A}$ for each $S \in \mathcal{C}(X)$.

▶ **Definition 43** (Functor $\mathcal{G}$). *We specify* $\mathcal{G} : \mathbf{QA}(\mathtt{QTh}_{CS}) \to \mathbf{EM}(\hat{\mathcal{C}})$ *as follows:*

- *on objects* $\mathbb{A} = (X, \Sigma^{\mathbb{A}}_{CS}, d)$, *we define* $\mathcal{G}(\mathbb{A}) = ((X, d), \alpha)$,
  with $\alpha : (\mathcal{C}(X), HK(d)) \to (X, d)$ *defined as:* $\alpha(S) = (\nu(S))^{\mathbb{A}}$,
- *on morphisms (i.e., non-expansive homomorphisms) we define* $\mathcal{G}(f) = f$.

In order to prove that $\mathcal{G}$ is well-defined on objects, we have to show that indeed $((X, d), \alpha)$ is an Eilenberg-Moore algebra for $\hat{\mathcal{C}}$, which amounts to proving the following lemma.

▶ **Lemma 44.** *Let* $\mathcal{G}(\mathbb{A}) = ((X, d), \alpha)$, *for* $\mathbb{A} = (X, \Sigma^{\mathbb{A}}_{CS}, d) \in \mathbf{QA}(\mathtt{QTh}_{CS})$.

1. (X, α) *is an Eilenberg-Moore algebra for* $\mathcal{C}$ *in* **Set**, *i.e.,* $\alpha \circ \eta^{\mathcal{C}} = id$ *and* $\alpha \circ \mathcal{C}\alpha = \alpha \circ \mu^{\mathcal{C}}$.
2. α *is a morphism in* **EMet**, *i.e.,* α *is a non-expansive map:* $d\langle\alpha, \alpha\rangle \sqsubseteq HK(d)$.

Proof. The proof of the first point does not have any specific metric–theoretic content and is omitted here. For the second point, let $S, T \in \mathcal{C}(X)$. By the definition of α, we have $d(\alpha(S), \alpha(T)) = d((\nu(S))^{\mathbb{A}}, (\nu(T))^{\mathbb{A}})$. As stated in Lemma 45 below, it is possible to derive in $\mathtt{QTh}_{CS}$ the quantitative inference

$$\bigcup_{(\Delta,\Theta)\in\mathtt{UB}(S)\times\mathtt{UB}(T)} \left(\bigcup_{(x,y)\in supp(\Delta)\times supp(\Theta)} \{x =_{d(x,y)} y\} \right) \vdash \nu(S) =_{HK(d)(S,T)} \nu(T)$$

which, since $\mathbb{A}$ is a model of $\mathtt{QTh}_{CS}$, is thereby satisfied by $\mathbb{A}$. Since all the premises of the inference hold in $\mathbb{A}$, we conclude that $d((\nu(S))^{\mathbb{A}}, (\nu(T))^{\mathbb{A}}) \leq HK(d)(S,T)$ and, therefore, $d\langle\alpha,\alpha\rangle \sqsubseteq HK(d)$ holds, as desired. ◀

The following technical lemma is critically used in the proof of Lemma 44(2) above. Note that its statement is purely syntactic as it deals with derivability in the deductive apparatus of quantitative equational theories (Definition 21).

▶ **Lemma 45.** *Let (X,d) be a metric space and let $S,T \in \mathcal{C}(X)$. Then we have in* $\mathtt{QTh}_{CS}$*:*

$$\bigcup_{(\Delta,\Theta)\in \mathtt{UB}(S)\times\mathtt{UB}(T)} \left(\bigcup_{(x,y)\in supp(\Delta)\times supp(\Theta)} \{x =_{d(x,y)} y\} \right) \vdash \nu(S) =_{HK(d)(S,T)} \nu(T)$$

Proof Sketch. First, we derive the following useful quantitative inference dealing with the case of $S = \{\Delta\}$ and $T = \{\Theta\}$ being singletons, so that $HK(d)(S,T) = K(d)(\Delta,\Theta)$. Let (X,d) be a metric space and let $\Delta,\Theta \in \mathcal{D}(X)$. Then the following is derivable in $\mathtt{QTh}_{CS}$:

$$\bigcup_{(x,y)\in supp(\Delta)\times supp(\Theta)} \{x =_{d(x,y)} y\} \vdash \nu(\{\Delta\}) =_{K(d)(\Delta,\Theta)} \nu(\{\Theta\}).$$

To construct this derivation we take an optimal coupling ω of Δ and Θ (see Definition 19) witnessing the Kantorovich distance $K(d)(\Delta,\Theta)$ and then use the information provided by ω to construct a syntactic derivation where only the quantitative inferences (A_p, C_p, I_p and K) of the quantitative theory of convex algebras are used. The construction of this derivation follows analogously to the completeness result for quantitative convex algebras from [36].

Secondly, we calculate the $HK(d)(S,T)$ distance between S and T.

$$HK(d)(S,T) = \max\big\{ \sup_{\Delta\in S} \inf_{\Theta\in T} K(d)(\Delta,\Theta) \quad , \quad \sup_{\Theta\in T} \inf_{\Delta\in S} K(d)(\Delta,\Theta) \big\}.$$

By compactness arguments, the inf and sup are always attained. Hence this calculation involves distances $K(d)(\Delta_i,\Theta_j)$ between a finite number of elements $\Delta_i \in S$ and $\Theta_j \in T$, for $0 \leq i \leq n$ and $0 \leq j \leq m$. Since the equation $x \oplus y = x \oplus y \oplus (x +_p y)$ holds in all convex semilattices, we can derive in the theory of convex semilattices the equalities: $\nu(S) = \nu(S) \oplus \nu(\{\Delta_1\}) \oplus \cdots \oplus \nu(\{\Delta_n\})$ and $\nu(T) = \nu(T) \oplus \nu(\{\Theta_1\}) \oplus \cdots \oplus \nu(\{\Theta_m\})$. For each of the pairs (Δ_i,Θ_j) appearing in the expressions above we can derive, as described above, the quantitative equation $\nu(\{\Delta_i\}) =_{K(d)(\Delta_i,\Theta_j)} \nu(\{\Theta_j\})$. The calculation of $HK(d)(S,T)$ can then be mimicked syntactically to derive the quantitative equation $\nu(S) =_{HK(d)(S,T)} \nu(T)$ by only using the quantitative inferences (A, C, I and H) of quantitative semilattices. This follows analogously to the completeness result for quantitative semilattices from [36]. ◀

It remains to verify that the functor $\mathcal{G}$ is well-defined on morphisms. To see this, take $f : X \to Y$ a non-expansive homomorphism of quantitative algebras $\mathbb{A} = (X, \Sigma^{\mathbb{A}}_{CS}, d)$ and $\mathbb{B} = (Y, \Sigma^{\mathbb{B}}_{CS}, d')$ in $\mathbf{QA}(\mathtt{QTh}_{CS})$. Then f is an arrow in $\mathbf{EMet}$, being non-expansive. We therefore only need to show that f is also a morphism of Eilenberg-Moore algebras (see Definition 6) i.e., that $f \circ \alpha = \beta \circ \hat{\mathcal{C}}(f)$. The verification of this equality involves no specific metric–theoretic considerations, and is therefore omitted.

5.3 The isomorphism

It remains to prove that the functors $\mathcal{F} : \mathbf{EM}(\hat{\mathcal{C}}) \to \mathbf{QA}(\mathtt{QTh}_{CS})$ and $\mathcal{G} : \mathbf{EM}(\hat{\mathcal{C}}) \to \mathbf{QA}(\mathtt{QTh}_{CS})$ define an isomorphism between the categories $\mathbf{EM}(\hat{\mathcal{C}})$ and $\mathbf{QA}(\mathtt{QTh}_{CS})$. This means proving that $\mathcal{G} \circ \mathcal{F} = id_{\mathbf{EM}(\hat{\mathcal{C}})}$ and $\mathcal{F} \circ \mathcal{G} = id_{\mathbf{QA}(\mathtt{QTh}_{CS})}$. On morphisms, by definition we have $\mathcal{G} \circ \mathcal{F}(f) = f = \mathcal{F} \circ \mathcal{G}(f)$. Hence the identities trivially hold true. The proofs regarding the identities on objects require only routine verifications, unfolding definitions, not involving any specific metric–theoretic content and therefore we omit them here.

6 Conclusions

We have introduced the **EMet** monad $\hat{\mathcal{C}}$ of finitely generated non–empty convex sets of distributions equipped with the Hausdorff-Kantorovich distance, and we have proved that $\hat{\mathcal{C}}$ is presented by the quantitative equational theory $\mathtt{QTh}_{CS}$ of quantitative convex semilattices. This result provides the basis for a foundational understanding of equational reasoning about program distances in processes combining nondeterminism and probabilities, as in bisimulation and trace metrics [22, 25, 26, 49, 6, 18]. This opens several directions for future research.

For instance, one interesting line of research is to examine the axiomatizations of bisimulation equivalences and metrics for nondeterministic and probabilistic programs (or process algebras) that have been proposed in the literature [40, 9, 21, 1, 2, 20]. The quantitative equational framework of quantitative convex semilattices provides a novel tool for comparing and further developing the existing works.

It is also important to explore variants of the **EMet** monad $\hat{\mathcal{C}}$ such as, for instance, the one that also includes the empty set. These are needed to model program observations such as termination. Following the ideas presented in [13], these variants can be explored via the lift monad $(\cdot + 1)$ and its quotients described by equational theories over the signature of convex semilattices extended with a new constant symbol. A systematic study of these quotients is a promising direction for future work. Applications to up-to techniques for bisimulation metrics [19, 10] could then be pursued as well.

Lastly, it is natural to ask if the monad $\hat{\mathcal{C}}$, and its presentation, can be obtained as a general categorical composition of the hyperspace monad $\mathcal{V}$ and the distribution monad $\hat{\mathcal{D}}$. Recently, Goy and Petrisan [30] have used the notion of weak distributive law to provide a positive answer for the corresponding monads in the category **Set**. Investigating whether this machinery is also applicable to the category **EMet** is an interesting topic for future work.

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